On P − reducible and R3 − Like – Generalized $\mathcal{B}$R − Recurrent Space

Fahmi Yaseen Abdo Qasem & Ala'a Abdalnasser Awad Abdallah
Dept. of Math., Faculty of Education-Aden, Univ. of Aden, Khormakssar, Aden, Yemen
fahmi.yaseen@yahoo.com, ala733.ala00@Gmail.com

Abstract
In this paper, we introduced the generalized $\mathcal{B}R$ − recurrent Finsler space, i.e. characterized by the following condition

$$\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh}) \quad , \quad R^i_{jkh} \neq 0 \ ^{*} ,$$

Where $\mathcal{B}_m$ is Berwald's covariant differential operator with respect to x^m , λ_m and μ_m are known as recurrence vectors .
The purpose of the present paper to develop the above space by study the properties of a P − reducible space and a $R3$ - like space, which their called a P − reducible generalized $\mathcal{B}R$ −recurrent space and a $R3$ – like generalized $\mathcal{B}R$ − recurrent space, respectively. Also to obtain different theorems for some tensors satisfy in above spaces. Various identities are established in our spaces.

Keywords: *a P − reducible generalized $\mathcal{B}R$ −recurrent space and a $R3$ – like generalized $\mathcal{B}R$ − recurrent space.*

1. Introduction

R. Verma [7] obtained the condition of a P − reducible R^h − recurrent space be a necessarily a Landsberg space.

* In Rund's book, R^i_{jkh} defined here, is defined by R^i_{hkj} . This difference must be noted.

M. Matsumoto [5] showed that the curvature tensor R_{ijkh} of a three dimensional Finsler space satisfies the condition

$$R_{ijkh} = g_{ik} L_{jh} + g_{jh} L_{ik} - k/h$$

and called it $R3$ − *like Finsler space*. Some properties of a $R3$ − like Finsler space were studied by H. Izumi and T.N. Srivastava [3] by introducing the idea of indicatorization. M. Yoshida [6] also discussed a $R3$ − like Finsler space and its special cases.

Let F_n be an n – dimensional Finsler space equipped with the metric function F(x,y) satisfying the request conditions [2] .

The vector y_i is defined by

(1.1) $$y_i = g_{ij}(x,y)y^j$$

The two sets of quantities g_{ij} and its associative g^{ij} , which are components of a metric tensor connected by

(1.2) $$g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 & if \ j = k , \\ 0 & if \ j \neq k . \end{cases}$$

In view of (1.1) and (1.2), we have

(1.3) a) $\delta^i_j g_{ir} = g_{jr}$, b) $\delta^i_j y^j = y^i$ and c) $\delta^i_j y_i = y_j$.

The tensor C_{ijk} is defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$$

which is positively homogeneous of degree -1 in y^i and symmetric in all its indices and called *(h)hv-torsion tensor* [4] and its associative C^i_{jk} is positively homogeneous of degree -1 in y^i and symmetric in its lower indices and called *(v)hv- torsion tensor*. According to Euler's theorem on homogeneous functions, these tensors satisfy the following:

(1.4) a) $C_{ijk}y^i = C_{kij}y^i = C_{jki}y^i = 0$ and b) $C^{\ i}_{jk}y^j = 0 = C^{\ i}_{kj}\ y^j$.

(1.5) a) $g_{ij}C^i_{kh} = C_{kjh}$ and b) $C^{\ i}_{jr}\,\delta^r_k = C^{\ i}_{jk}$.

Berwald covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is given by

$$\mathcal{B}_k T^i_j \coloneqq \partial_k T^i_j - (\dot{\partial}_r T^i_j)G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk} \ .$$

Berwald covariant derivative of the vector y^i vanish identically, i.e.

(1.6) $\mathcal{B}_k y^i = 0$

In general, Berwald covariant derivative of the metric tensor g_{ij} doesn't vanish and given by

(1.7) $\mathcal{B}_k g_{ij} = -2C_{ijk|h}y^h = -2y^h\mathcal{B}_h C_{ijk}$.

The tensor R^i_{jkh} is called *Cartan's third curvature tensor* is defined by

$$R^i_{jkh} = \partial_h \Gamma^{*i}_{jk} + (\partial_l \Gamma^{*i}_{jk})G^l_h + C^i_{jm}(\partial_k G^m_h - G^m_{kl}G^l_h) + \Gamma^{*i}_{mk}\Gamma^{*m}_{jh} - k/h^{\,*} \ .$$

Ricci tensor R_{jk} of the curvature tensor R^i_{jkh} is given by

(1.8) $R^i_{jki} = R_{jk}$.

The associative curvature tensor R_{pjkh} satisfies

(1.9) $R_{pjkh} = g_{pi}R^i_{jkh}$.

The tensor K^i_{jkh} is called *Cartan's is fourth curvature tensor* defined as follows:

$$K^i_{jkh} \coloneqq \partial_h \Gamma^{*i}_{kj} + (\dot{\partial}_l \Gamma^{*i}_{jh})G^l_k + \Gamma^{*i}_{mh}\Gamma^{*m}_{kj} - h/k \ .$$

This curvature tensor K^i_{jkh} is positively homogeneous of degree zero in y^i and skew-symmetric in its last two lower indices.

Ricci tensor K_{jk} of the curvature tensor K^i_{jkh} is given by

(1.10) $K^i_{jki} = K_{jk}$.

* - h /k means the subtraction from the former term by interchanging the indices h and k .

The associative curvature tensor K_{pjkh} satisfies

(1.11) $K_{pjkh} = g_{pi}K^i_{jkh}$.

We know that [2]

(1.12) $R^i_{jkh} = K^i_{jkh} + C^i_{js}H^s_{kh}$

and

(1.13) $R_{ijkh} = K_{ijkh} + C_{ijs}K^s_{kh}$.

The curvature tensor R^i_{jkh} and the h(v)- torsion tensor H^i_{kh} are connected by

(1.14) $R^i_{jkh}y^j = H^i_{kh} = K^i_{jkh}y^j$.

The hv- curvature tensor $P^{\ i}_{jkh}$ is positively homogeneous of degree zero in y^i and satisfies the relation

(1.15) $P^{\ i}_{jkh}\,y^j = \Gamma^{*i}_{hjk}y^j = P^{\ i}_{kh} = C^{\ i}_{kh|r}y^r$.

The curvature vector P_k is given by

(1.16) $P^{\ i}_{ki} = P_k$.

2. A Generalized $\mathcal{BR}-$ Recurrent Space

Let us consider a Finsler space F_n for which Cartan's third curvature tensor R^i_{jkh} satisfies the generalized recurrence property with respect to Berwald's connection parameter G^i_{kh} , i.e. characterized by the following condition

(2.1) $$\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh}) \quad , \quad R^i_{jkh} \neq 0 \ ,$$

where $\mathcal{B}_m$ is Berwald's covariant differential operator with respect to x^m , λ_m and μ_m are called *recurrence vectors* .

Definition 2.1. A Finsler space F_n for which Cartan's third curvature tensor R^i_{jkh} satisfies the condition (2.1), where λ_m and μ_m are non-zero covariant vectors field. Such space and the tensor which satisfy the condition (2.1) will celled a *generalized* $\mathcal{BR}-$*recurrent space* and a *generalized* $\mathcal{B}-$*recurrent tensor*, respectively and denoted them briefly by G$(\mathcal{BR})-RF_n$ and G$\mathcal{B}-R$, respectively .

Transvecting the condition

(2.2) $$\mathcal{B}_m P^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh}) \ , \{ (2.17), [1] \}$$

by y^j using (1.15), (1.6), (1.3b) and (1.1), we get

(2.3) $$\mathcal{B}_m P^i_{kh} = \lambda_m P^i_{kh} + \mu_m(y^i g_{kh} - \delta^i_k y_h) \ .$$

Contracting the indices i and h in the condition (2.3), using (1.16), (1.1) and (1.3c), we get

(2.4) $$\mathcal{B}_m P_k = \lambda_m P_k \ .$$

Transvecting the condition (2.1) by y^j , using (1.14), (1.6), (1.3b) and (1.1), we get

(2.5) $$\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m(y^i g_{kh} - \delta^i_k y_h) \ .$$

The curvature tensors R^i_{jkh} and K^i_{jkh} are related by the formula (1.12) [2] . Taking the covariant derivative for (1.12) with respect to x^m in the sense of Berwald, we get

(2.6) $$\mathcal{B}_m R^i_{jkh} = \mathcal{B}_m K^i_{jkh} + (\mathcal{B}_m C^i_{jr})H^r_{kh} + C^i_{jr}(\mathcal{B}_m H^r_{kh}) \ .$$

Using the conditions (2.1) and (2.5) in (2.6), we get

(2.7) $$\lambda_m R^i_{jkh} + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh}) = \mathcal{B}_m K^i_{jkh} + (\mathcal{B}_m C^i_{jr})H^r_{kh} + C^i_{jr}\,[\,\lambda_m H^r_{kh} + \mu_m(y^r g_{kh} - \delta^r_k y_h)\,] \ .$$

Using (1.12), (1.4b) and (1.5b) in (2.7), we get

(2.8) $$\mathcal{B}_m K^i_{jkh} = \lambda_m K^i_{jkh} + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh}) - (\mathcal{B}_m C^i_{jr})H^r_{kh} + \mu_m C^i_{jk} y_h \ .$$

This shows that

(2.9) $$\mathcal{B}_m K^i_{jkh} = \lambda_m K^i_{jkh} + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh})$$

if and only if

(2.10) $$(\mathcal{B}_m C^i_{jr})H^r_{kh} - \mu_m C^i_{jk} y_h = 0 \ .$$

3. $P-$ Reducible $-$Generalized $\mathcal{BR}-$ Recurrent Space

A $P-$reducible space is characterized by the condition [(3.1)

$$P_{jkh} = \tfrac{1}{n-1}(P_j h_{kh} + P_k h_{hj} + P_h h_{jk}) \ ,$$

Where $P_{jkh} = C_{jkh|m} y^m$, $P^i_{ik} = P_k$ and $h_{ij} := g_{ij} - l_i l_j$.

Definition 3.1. The generalized $\mathcal{BR}-$ recurrent space which is a $P-$*reducible* space [satisfies the condition (3.1)], will be called a $P-$*reducible generalized* $\mathcal{BR}-$*recurrent space* and will denote it briefly by $P-reducible-G(\mathcal{BR})-RF_n$.

Let us consider a $\mathrm{P}-$reducible $-\mathrm{G}(\mathcal{B}\mathrm{R})-\mathrm{RF_n}$.

By using (1.9) and (1.5a) in (1.13), we get

(3.2) $$K_{ijkh} = g_{rj}R^r_{ikh} - g_{rj}C^r_{is}H^s_{kh} .$$

Taking the covariant derivative for (3.2) with respect to x^m in the sense of Berwald, we get

$$\mathcal{B}_m K_{ijkh} = \mathcal{B}_m(g_{rj}R^r_{ikh}) - \mathcal{B}_m(g_{rj}C^r_{is}H^s_{kh})$$

or

(3.3) $$\mathcal{B}_m K_{ijkh} = R^r_{ikh}\mathcal{B}_m g_{rj} + g_{rj}\mathcal{B}_m R^r_{ikh} - C^r_{is}H^s_{kh}\mathcal{B}_m g_{rj} - g_{rj}\mathcal{B}_m C^r_{is}H^s_{kh} .$$

Using (1.7) and the condition (2.1) in (3.3), we get

$$\mathcal{B}_m K_{ijkh} = R^r_{ikh}(-2C_{rjm|h}y^h) + g_{rj}[\,\lambda_m R^r_{ikh} + \mu_m(\delta^r_i g_{kh} - \delta^r_k g_{ih})\,] - C^r_{is}H^s_{kh}(-2C_{rjm|h}y^h) - g_{rj}\mathcal{B}_m C^r_{is}H^s_{kh} .$$

or

(3.4) $$\mathcal{B}_m K_{ijkh} = (R^r_{ikh} - C^r_{is}H^s_{kh})(-2C_{rjm|h}y^h) + \lambda_m g_{rj}R^r_{ikh} + \mu_m g_{rj}(\delta^r_i g_{kh} - \delta^r_k g_{ih}) - g_{rj}\mathcal{B}_m C^r_{is}H^s_{kh} .$$

Using (1.12) and the condition (3.1) in (3.4), we get

(3.5) $$\mathcal{B}_m K_{ijkh} = K^r_{ikh}\left[\frac{-2}{n-1}(P_r h_{jm} + P_j h_{mr} + P_m h_{rj})\right] + \lambda_m g_{rj}(\,K^r_{ikh} + C^r_{is}H^s_{kh}) + \mu_m g_{rj}(\delta^r_i g_{kh} - \delta^r_k g_{ih}) - g_{rj}\mathcal{B}_m C^r_{is}H^s_{kh} .$$

This shows that

(3.6) $$\mathcal{B}_m(C^r_{is}H^s_{kh}) = \lambda_m(C^r_{is}H^s_{kh}) + \mu_m(\delta^r_i g_{kh} - \delta^r_k g_{ih})$$

if and only if

(3.7) $$\mathcal{B}_m K_{ijkh} = K^r_{ikh}\left[\lambda_m g_{rj} - \frac{2}{n-1}(P_r h_{jm} + P_j h_{mr} + P_m h_{rj})\right],$$

since $g_{rj} \neq 0$.

Thus, we conclude

Theorem 3.1. *In $P-reducible-G(\mathcal{B}R)-RF_n$, the tensor $(C^r_{is}H^s_{kh})$ is a generalized recurrent if and only if (3.7) holds good .*

By using (1.11), (1.5a), (1.3a) and (1.7) in (3.5), we get

(3.8) $$\mathcal{B}_m K_{ijkh} = \frac{-2}{n-1}K^r_{ikh}\big(P_r h_{jm} + P_j h_{mr} + P_m h_{rj}\big) + \lambda_m\big(K_{ijkh} + C_{ijs}H^s_{kh}\big) + \mu_m\big(g_{ij}g_{kh} - g_{kj}g_{ih}\big) - \mathcal{B}_m C_{ijs}H^s_{kh} - 2\,(C^r_{is}H^s_{kh})C_{rjm|h}y^h,$$

where $P_{rjm} = C_{rjm|h}y^h$.

Using (3.1) in (3.8), we get

(3.9) $$\mathcal{B}_m K_{ijkh} = \frac{-2}{n-1}(K^r_{ikh} + C^r_{is}H^s_{kh})\big(P_r h_{jm} + P_j h_{mr} + P_m h_{rj}\big) + \lambda_m(K_{ijkh} + C_{ijs}H^s_{kh}) + \mu_m(g_{ij}g_{kh} - g_{kj}g_{ih}) - \mathcal{B}_m C_{ijs}H^s_{kh}.$$

Using (1.12) in (3.9), we get

$$\mathcal{B}_m K_{ijkh} = \frac{-2}{n-1}R^r_{ikh}(P_r h_{jm} + P_j h_{mr} + P_m h_{rj}) + \lambda_m(K_{ijkh} + C_{ijs}H^s_{kh}) + \mu_m(g_{ij}g_{kh} - g_{kj}g_{ih}) - \mathcal{B}_m C_{ijs}H^s_{kh} .$$

This shows that

(3.10) $$\mathcal{B}_m(C_{ijs}H^s_{kh}) = \lambda_m(C_{ijs}H^s_{kh}) + \mu_m(g_{ij}g_{kh} - g_{kj}g_{ih})$$

if and only if

(3.11) $$\mathcal{B}_m K_{ijkh} = \lambda_m K_{ijkh} - \frac{2}{n-1}R^r_{ikh}(P_r h_{jm} + P_j h_{mr} + P_m h_{rj}) .$$

Thus, we conclude

Theorem 3.2. *In $P-reducible-G(\mathcal{B}R)-RF_n$, the tensor $(C_{ijs}H^s_{kh})$ is a generalized recurrent if and only if (3.11) holds good .*

By using (1.11), the equation (3.7) becomes

(3.12) $\mathcal{B}_m K_{ijkh} = \lambda_m K_{ijkh} - \frac{2}{n-1} K^r_{ikh}(P_r h_{jm} + P_j h_{mr} + P_m h_{rj})$

This shows that

(3.13) $\mathcal{B}_m K_{ijkh} = \lambda_m K_{ijkh}$

if and only if

(3.14) $K^r_{ikh}(P_r h_{jm} + P_j h_{mr} + P_m h_{rj}) = 0$.

Thus, we conclude

Theorem 3.3. *In $P - reducible - G(\mathcal{B}R) - RF_n$, we have the identity (3.14) if and only if the associative curvature tensor K_{ijkh} is recurrent .*

Taking the covariant derivative for (3.1) with respect to x^m in the sense of Berwald, we get

(3.15) $$\mathcal{B}_m P_{jkh} = \frac{1}{n-1} \left(h_{kh}\mathcal{B}_m P_j + h_{hj}\mathcal{B}_m P_k + h_{jk}\mathcal{B}_m P_h + P_j \mathcal{B}_m h_{kh} + P_k \mathcal{B}_m h_{hj} + P_h \mathcal{B}_m h_{jk}\right) .$$

Using (2.4) in (3.15), we get

(3.16) $$\mathcal{B}_m P_{jkh} = \frac{\lambda_m}{n-1} \left(P_j h_{kh} + P_k h_{hj} + P_h h_{jk}\right) + \frac{1}{n-1} \left(P_j \mathcal{B}_m h_{kh} + P_k \mathcal{B}_m h_{hj} + P_h \mathcal{B}_m h_{jk}\right) .$$

Using (3.1) in (3.16), we get

$$\mathcal{B}_m P_{jkh} = \lambda_m P_{jkh} + \frac{1}{n-1} \left(P_j \mathcal{B}_m h_{kh} + P_k \mathcal{B}_m h_{hj} + P_h \mathcal{B}_m h_{jk}\right) .$$

This shows that

(3.17) $\mathcal{B}_m P_{jkh} = \lambda_m P_{jkh}$

if and only if

(3.18) $P_j \mathcal{B}_m h_{kh} + P_k \mathcal{B}_m h_{hj} + P_h \mathcal{B}_m h_{jk} = 0$.

Thus, we conclude

Theorem 3.4. *In $P - reducible - G(\mathcal{B}R) - RF_n$, we have the identity (3.18) if and only if the associative torsion tensor P_{jkh} is recurrent .*

Using the condition (3.1) in (3.11), we get

$$\mathcal{B}_m K_{ijkh} = \lambda_m K_{ijkh} - 2R^r_{ikh} P_{rjm} .$$

This shows that

(3.19) $\mathcal{B}_m K_{ijkh} = \lambda_m K_{ijkh}$

if and only if

(3.20) $-2\, P_{rjm} = 0$.

Thus, we conclude

Theorem 3.5. *The $P - reducible - G(\mathcal{B}R) - RF_n$ is Landsberg space if and only if the associative curvature tensor K_{ijkh} is recurrent .*

4. $R3 -$ Like $-$ Generalized $\mathcal{B}R -$ Recurrent Space

The curvature tensor R_{ijhk} of a three dimensional Finsler space of the form [5]

(4.1) $R_{ijkh} = g_{ik} L_{jh} + g_{jh} L_{ik} - k/h$,

where

(4.2) $L_{ik} = \frac{1}{n-2} \left(R_{ik} - \frac{r}{2} g_{ik} \right)$,

$R_{jk} := R^i_{jki}$

and

(4.3) $r = \frac{1}{n-1} R^i_i$.

Definition 4.1. The generalized $\mathcal{B}R -$ recurrent space which is a $R3 -$ Like space [satisfies the condition (4.1)], will be called a $R3 -$ *Like generalized* $\mathcal{B}R -$ *recurrent space* and will denote it briefly by $R3 - Like - G(\mathcal{B}R) - RF_n$.

M. Matsumoto [5] introduced a Finsler space F_n $(n > 3)$ for the associative of Cartan's second curvature tensor R_{ijkh} satisfies the above condition and called it $R3$ - *like Finsler space* .

Let us consider a $\mathrm{R3 - Like - G(\mathcal{B}R) - RF_n}$.

Using (1.9) in the condition (4.1), we get

(4.4) $\quad g_{rj}R^r_{ikh} = g_{ik}\ L_{jh} + g_{jh}\ L_{ik} - k/h$.

Taking the covariant derivative for (4.4) with respect to x^m in the sense of Berwald and using the condition (3.1), we get

(4.5) $\quad g_{rj}\,\mathcal{B}_m R^r_{ikh} + R^r_{ikh}\,\mathcal{B}_m g_{rj} = \mathcal{B}_m R_{ijkh}$.

Using (1.12) in (4.5), we get

(4.6) $\quad g_{rj}\big[\,\mathcal{B}_m K^r_{ikh} + \mathcal{B}_m(C^r_{is}H^s_{kh}\,)\,\big] + R^r_{ikh}\,\mathcal{B}_m g_{rj} = \mathcal{B}_m R_{ijkh}$.

Using (2.9) in (4.6), we get

(4.7) $\quad g_{rj}[\,\lambda_m K^r_{ikh} + \mu_m(\delta^r_i\, g_{kh} - \delta^r_k g_{ih})\,] + g_{rj}\mathcal{B}_m(C^r_{is}H^s_{kh}) + R^r_{ikh}\,\mathcal{B}_m g_{rj}$
$= \mathcal{B}_m R_{ijkh}$.

Using (1.12) and (1.9) in (4.7), we get

(4.8) $\quad \left[\lambda_m g_{rj}R^r_{ikh} - \lambda_m g_{rj}C^r_{is}H^s_{kh} + \mu_m g_{rj}(\delta^r_i\, g_{kh} - \delta^r_k g_{ih})\right] +$
$g_{rj}\mathcal{B}_m C^r_{is}H^s_{kh} + R^r_{ikh}\,\mathcal{B}_m g_{rj} = g_{rj}\mathcal{B}_m R^r_{ikh} + R^r_{ikh}\mathcal{B}_m g_{rj}$.

This shows that

(4.9) $\quad \mathcal{B}_m(C^r_{is}H^s_{kh}) = \lambda_m(C^r_{is}H^s_{kh})$

if and only if

(4.10) $\quad \mathcal{B}_m R^r_{ikh} = \lambda_m R^r_{ikh} + \mu_m(\delta^r_i\, g_{kh} - \delta^r_k g_{ih})$.

Thus, we conclude

Theorem 4.1. *In* $R3 - Like - G(\mathcal{B}R) - RF_n$ *, the tensor* $(\,C^r_{is}H^s_{kh}\,)$ *behaves as recurrent if and only if the curvature tensor* R^r_{ikh} *is generalized recurrent [provided (2.10) holds] .*

Using (1.11) in (1.13), we get

(4.11) $\quad R_{ijkh} = g_{rj}K^r_{ikh} + C_{ijs}H^s_{kh}$.

Taking the covariant derivative for (4.11) with respect to x^m in the sense of Berwald, we get

(4.12) $\quad \mathcal{B}_m R_{ijkh} = g_{rj}\mathcal{B}_m K^r_{ikh} + K^r_{ikh}\mathcal{B}_m g_{rj} + \mathcal{B}_m C_{ijs}H^s_{kh}$.

Using (2.9) in (4.12), we get

(4.13) $\quad \mathcal{B}_m R_{ijkh} = g_{rj}[\,\lambda_m K^r_{ikh} + \mu_m(\delta^r_i\, g_{kh} - \delta^r_k g_{ih})\,] + K^r_{ikh}\mathcal{B}_m g_{rj}$
$+ \mathcal{B}_m C_{ijs}H^s_{kh}$.

Using (1.12) and (4.1) in (4.13), we get

(4.14) $\quad \mathcal{B}_m(\,g_{ik}\ L_{jh} + g_{jh}\ L_{ik} - k/h) = \lambda_m g_{rj}R^r_{ikh} - \lambda_m g_{rj}C^r_{is}H^s_{kh}$
$+ \mu_m g_{rj}(\delta^r_i\, g_{kh} - \delta^r_k g_{ih}) + K^r_{ikh}\mathcal{B}_m g_{rj} + \mathcal{B}_m C_{ijs}H^s_{kh}$.

Using (1.9), (4.1) and (1.3a) in (4.14), we get

(4.15) $\quad \mathcal{B}_m(\,g_{ik}\ L_{jh} + g_{jh}\ L_{ik} - k/h) = \lambda_m(\,g_{ik}\ L_{jh} + g_{jh}\ L_{ik} - k/h) -$
$\lambda_m g_{rj}C^r_{is}H^s_{kh} + \mu_m(g_{ij}g_{kh} - g_{kj}g_{ih}) + K^r_{ikh}\mathcal{B}_m g_{rj} + \mathcal{B}_m(C_{ijs}H^s_{kh})$.

This shows that

(4.16) $\quad \mathcal{B}_m(\,g_{ik}\ L_{jh} + g_{jh}\ L_{ik} - k/h) = \lambda_m(\,g_{ik}\ L_{jh} + g_{jh}\ L_{ik} - k/h)$
$+ \mu_m(g_{ij}g_{kh} - g_{kj}g_{ih})$

if and only if

(4.17) $K_{ikh}^{r}\mathcal{B}_m g_{rj} + \mathcal{B}_m(C_{ijs}H_{kh}^{s}) = \lambda_m g_{rj} C_{ijs} H_{kh}^{s}$.

Thus, we conclude

Theorem 4.2. *In* $3 - Like - G(\mathcal{B}R) - RF_n$ *, the tensor* $(g_{ik}\ L_{jh} + g_{jh}\ L_{ik} - k/h)$ *is a generalized recurrent if and only if (4.17) holds good [provided (2.10) holds].*

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